On complete systems of automata

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It was shown in [6] that there exists no finite complete system of finite automata with respect to R-product. In this paper we show that there exists no system of finite automata which is minimal and isomorphically complete with respect to quasidirect product. Analogous statements are valid with respect to quasi-superposition and R-product.

In order to investigate these questions we need some notions and notations (see also [1]–[4]).

A system $\mathfrak{A}$ of Mealy automata[1]) is said to be *isomorphically complete* with respect to R-product (quasi-direct product, quasi-superposition) if for every automaton $\mathbf{A}$ an R-product (quasi-direct product, quasi-superposition) $\mathbf{B}$ of automata from $\mathfrak{A}$ can be found such that $\mathbf{A}$ is A-isomorphic to some A-subautomaton of $\mathbf{B}$.

Let $\mathbf{A}=\mathbf{A}(X, A, Y, \delta, \lambda)$ be an arbitrary automaton. A partition π of A into disjoint subsets is called a *congruent partition* of $\mathbf{A}$ if for every $a, b \in A$ and $x \in X$

$$a \equiv b(\pi) \Rightarrow \delta(a, x) \equiv \delta(b, x)\ (\pi)$$

holds (see [5]).

An arbitrary automaton $\mathbf{A}=\mathbf{A}(X, A, Y, \delta, \lambda)$ is said to be an *x-prime automaton* ($x \in X$) if for some $a \in A$ the number of states of the automata $\mathbf{A}$ and $\mathbf{A}^{(a,x)}$ (see [7]) is the same prime number p and there exists a permutation

$$\begin{pmatrix} 1 \cdots p \\ 1' \cdots p' \end{pmatrix}$$

such that

$$\delta(a_{i'}, x) = \begin{cases} a_{(i+1)'} & \text{if } i' < p' \\ a_{1'} & \text{if } i' = p' \end{cases} \qquad (A = \langle a_1, \ldots, a_p \rangle).$$

It follows from the proof of Lemma and Theorem of [7] that if the automaton $\mathbf{A}=\mathbf{A}(X, A, Y, \delta, \lambda)$ is x-prime for some $x \in X$, then $\mathbf{A}$ has only trivial partitions.

[1]) By "automaton" we always mean a finite automaton.

We require the following lemmas:

Lemma 1. *If $\mathfrak{A}=\langle A_1, \mathbf{A}_2, \ldots\rangle$ is a system of automata which is isomorphically complete with respect to quasi-direct product and if for some i, j $(i\neq j)$ the automaton A_i^* can be isomorphically embedded into $\mathbf{A}_j^*$,*[2] *then the system $\mathfrak{A}'=\mathfrak{A}\setminus\langle \mathbf{A}_i\rangle$ is also isomorphically complete with respect to quasi-direct product.*

The statements of Lemma 1 are trivial.

Lemma 2. *If an x-prime automaton $\mathbf{A}$ with reduced inputs*[3] *can be A-isomorphically embedded into some quasi-direct product*

$$\prod_{i=1}^{k} \mathbf{A}_i[X, Y, \varphi, \psi]$$

of automata $\mathbf{A}_i=\mathbf{A}_i(X_i, A_i, Y_i, \delta_i, \lambda_i)$ $(i=1, \ldots, k)$, then for some i $(1\leqq i\leqq k)$ the automaton $\mathbf{A}^*$ can be (X, A)-isomorphically embedded into $\mathbf{A}_i^*$.

Proof. Suppose the condition of Lemma 2 is true. Then,

$$a=(a_1, \ldots, a_k) \qquad (a_i\in A_i;\ i=1, \ldots, k)$$

holds for arbitrary $a\in A$. The following partitions π_i $(i=1, \ldots, k)$ are congruent:

$$a(=(a_1, \ldots, a_k)) \equiv ((a_1', \ldots, a_k') =)a'(\pi_i) \Leftrightarrow a_i = a_i',$$

because by virtue of the definition of quasi-direct product we have

$$\varphi(a, x) = (\varphi_1(x), \ldots, \varphi_i(x), \ldots, \varphi_k(x))$$

for arbitrary $x\in X$.

Since $\mathbf{A}$ is an x-prime automaton, it has only trivial partitions. From this it follows that there exists an i $(1\leqq i\leqq k)$ such that for arbitrary $a, a'\in A$

$$(1) \qquad a(=(a_1, \ldots, a_i, \ldots, a_k)) \neq ((a_1', \ldots, a_i', \ldots, a_k') =)a' \Rightarrow a_i\neq a_i'.$$

Consider the maps $\varrho_1: X\to X_i$ and $\varrho_2: A\to A_i$ defined by

$$\varrho_1(x)=\varphi_i(x) \quad \text{and} \quad \varrho_2(a)=a_i,$$

where $a=(a_1, \ldots, a_i, \ldots, a_k)$.

It is clear that, by virtue of (1), the map ϱ_2 is one-to-one. We show that ϱ_1 and ϱ_2 are homomorphisms.

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[2]) A^* is the Medvedev automaton obtained from A in a natural way (see [1]).

[3]) An automaton with reduced input is defined to be an automaton such that different inputs will induce different maps of the set of states.

Take $a=(a_2, \ldots, a_i, \ldots, a_k)$, that is, $\varrho_2(a)=a_i$ and $\varrho_1(x)=\varphi_i(x)=x_i$. Then

$$\varrho_2(\delta(a, x)) = \varrho_2(\delta_1(a_1, x_1), \ldots, \delta_i(a_i, x_i), \ldots, \delta_k(a_k, x_k)) =$$

$$= \varrho_2\big(\delta_1(a_1, \varphi_1(x)), \ldots, \delta_i(\varrho_2(a), \varrho_1(x)), \ldots, \delta_k(a_k, \varphi_k(x))\big) = \delta_i(\varrho_2(a), \varrho_1(x)).$$

The map ϱ_1 is one-to-one since $\mathbf{A}$ is an automaton with reduced input. This proves that the pair (ϱ_1, ϱ_2) is an (X, A)-isomorphism of $\mathbf{A}^*$ into A_i^*. This completes the proof of Lemma 2.

Using Lemma 1 and 2, we can prove:

Theorem 1. *There is no system of automata which is minimal and isomorphically complete with respect to quasi-direct product.*

Proof. Let $\mathfrak{A}=\langle \mathbf{A}_i=\mathbf{A}_i(X_i, A_i, Y_i, \delta_i, \lambda_i) | i=1, 2, \ldots\rangle$ be an arbitrary system of automata, which is isomorphically complete with respect to quasi-direct product. We have to show that for arbitrary $\mathbf{A}_i$ the system $\mathfrak{A}'=\mathfrak{A}\setminus\langle \mathbf{A}_i\rangle$ is also isomorphically complete with respect to quasi-direct product. This already gives the proof of Theorem 1.

Let $\mathbf{A}=\mathbf{A}(X, Y, A, \delta, \lambda)$ be an automaton with reduced inputs which is an x-prime automaton for some $x\in X$ and such that the automaton $\mathbf{A}_i^*$ can be embedded (X, A)-isomorphically into $\mathbf{A}^*$, furthermore let $\bar{\bar{\mathbf{A}}} > \bar{\bar{\mathbf{A}}}_i$. Such an automaton $\mathbf{A}$ exists. Indeed, let p be a prime number such that $p-\bar{\bar{A}}_i > \bar{\bar{X}}_{i+1}$. Let us add to the set $A_i=\langle a_{i1}, \ldots, a_{ik}\rangle$ the elements $a_{i,k+1}, \ldots, a_{ip}$, and to the set X_i of inputs a symbol x which is not in X_i. We extend the state transition function δ_i for $A=\langle a_{i1}, \ldots, a_{ik}, a_{ik+1}, \ldots, a_{ip}\rangle$ and for $X=X_i\cup\langle x\rangle$ in the following way:

$$\delta_i(a_{il}, x) = \begin{cases} a_{il+1} & \text{if } \ i_l < p, \\ a_{i1} & \text{if } \ i_l = p, \end{cases}$$

and let $\delta_i(a_{il}, x_i)$ $(x_i\in X_i)$ be an arbitrary element of A such that $\mathbf{A}$ is an automaton with reduced input. This is possible since $p-\bar{\bar{\mathbf{A}}}_i > X_{i+1}$. The function λ can be chosen arbitrarily.

Because the system $\mathfrak{A}$ is isomorphically complete with respect to quasi-direct product, the automaton $\mathbf{A}$ can be embedded A-isomorphically in a quasi-direct product of automata from $\mathfrak{A}$. In this way, by Lemma 2, the automaton $\mathbf{A}^*$ can be embedded (X, A)-isomorphically in an automaton $\mathbf{A}_j^*$ $(\mathbf{A}_j\in\mathfrak{A})$. We have $j\neq i$ because $\bar{\bar{A}} > \bar{\bar{A}}_i$. Since $\mathbf{A}_i^*$ can be embedded (X, A)-isomorphically in $\mathbf{A}^*$, it can be embedded (X, A)-isomorphically in $\mathbf{A}_j^*$, that is, by Lemma 1, $\mathfrak{A}'=\mathfrak{A}\setminus\langle \mathbf{A}_i\rangle$ is an isomorphically complete system with respect to quasi-direct product.

We now prove that there exists no system of automata which is minimal and isomorphically complete with respect to quasi-superposition. In order to prove this statement we need the following trivial

Lemma 3. *If $\mathfrak{A}=\langle \mathbf{A}_1, \mathbf{A}_2, \ldots\rangle$ is a system of automata which is isomorphically complete with respect to quasi-superposition and if for some i, j $(i\neq j)$ the automaton $\mathbf{A}_i^*$ can be isomorphically embedded into $\mathbf{A}_j^*$, then the system $\mathfrak{A}'=\mathfrak{A}\setminus\langle \mathbf{A}_i\rangle$ is also isomorphically complete with respect to quasi-superposition.*

We also need the following

Lemma 4. *If an x-prime automaton $\mathbf{A}=\mathbf{A}(X, A, Y, \delta, \lambda)$ with reduced input can be A-isomorphically embedded into some quasi-superposition $\mathbf{A}_1^{(\gamma_1, \lambda_1)} \ldots \mathbf{A}_k^{(\gamma_k, \lambda_k)}$ of automata $\mathbf{A}_i=\mathbf{A}_i(X_i, A_i, Y_i, \delta_i, \lambda_i)$ $(i=1, \ldots, k)$, then for some $i\,(1\leqq i\leqq k)$ the automaton $\mathbf{A}^*$ can be (X, A)-isomorphically embedded into $\mathbf{A}_i^{(\gamma_i, \lambda_i)*}$.*

Proof. Under the condition of the lemma, we have $a=(a_1, \ldots, a_k)$ for an arbitrary $a\in A$.

It is obvious that the following partitions π_i $(i=1, \ldots, k)$ of $\mathbf{A}$ are congruent:

$$a(=(a_1, \ldots, a_i, \ldots, a_k)) \equiv ((a_1', \ldots, a_i', \ldots, a_k')=)a'(\pi_i) \Leftrightarrow a_j = a_j' \qquad (j=1, \ldots, i).$$

Since $\mathbf{A}$ is an x-prime automaton so it has only trivial partitions. From this it follows that there is an $i\,(1\leqq i\leqq k)$ such that for arbitrary $a, a'\in A$ we have

$$a(=(a_1, \ldots, a_i, \ldots, a_k)) \neq ((a_1', \ldots, a_i', \ldots, a_k')=)a' \Rightarrow a_i \neq a_i'. \tag{2}$$

Let j be the minimum of the numbers i satisfying (2). If $j\geqq 2$, then

$$a_1 = a_1', \ldots, a_{j-1} = a_{j-1}'$$

hold for $l=1, \ldots, j-1$ and arbitrary $a, a'\in A$.

We now show that $\mathbf{A}^*$ can be (X, A)-isomorphically embedded in to $\mathbf{A}_j^{(\gamma_j, \lambda_j)*}$. A suitable (X, A)-isomorphism (ϱ_1, ϱ_2) is the following:

$$\varrho_1(x) = \lambda_j(a_{j-1}, \gamma_{j-1}(\ldots, \gamma_2(\lambda_1(a_1, \gamma_1(x)))\ldots)), \quad \varrho_2(a) = a_j,$$

where a_m $(m=1, 2, \ldots, j)$ is the mth component of a.

The map ϱ_1 is one-to-one since $\mathbf{A}$ is an automaton with reduced inputs. It follows from (2) that ϱ_2 is a 1—1 map. Since, by the choice of j,

$$\delta_l(a_l, \gamma_l(\lambda_{l-1}(a_{l-1}, \gamma_{l-1}(\ldots, \gamma_2(\lambda_1(a_1, \gamma_1(x)))\ldots)))) = a_l$$

holds for each $x\in X$ and $l\,(1\leqq l\leqq j-1)$, where $a_1, \ldots, a_{l-1}, a_l$ are components of a, the pair (ϱ_1, ϱ_2) is an (X, A)-isomorphism.

It is clear that if $j=1$ then the isomorphism (ϱ_1, ϱ_2) can be given in the form

$$\varrho_1(x)=x \quad\text{and}\quad \varrho_2(a)=a_1,$$

where a_1 is the first component of a. This concludes the proof of Lemma 4.

If we repeat the proof of Theorem 1, using Lemma 3 and 4 instead of Lemma 1 and 2, respectively, and the quasi-superposition instead of quasi-direct product, we get the following

Theorem 2. *There exists no system of automata which is minimal and isomorphically complete with respect to quasi-superposition.*

We also have the following

Theorem 3. *There is no system of automata which is minimal and isomorphically complete with respect to R-product.*

Theorem 3 obviously follows from Theorem 2 and the following

Theorem 4. *An automaton* $\mathbf{A}=\mathbf{A}(X, A, Y, \delta, \lambda)$ *is an R-product of automata* $\mathbf{A}_i=\mathbf{A}_i(X'_i, A_i, Y'_i, \delta'_i, \lambda'_i)$ $(i=1, \ldots, k)$ *if and only if* $\mathbf{A}$ *is a quasi-superposition of the same automata* $\mathbf{A}_i$ $(i=1, \ldots, k)$.

Proof. The necessity is obvious because quasi-superpositions are special cases of R-products.

Conversely, let $\mathbf{A}$ be an R-product of automata A_i $(i=1, \ldots, k)$, that is,

$$\mathbf{A} = \prod_{i=1}^{k} \mathbf{A}_i[X, Y, \varphi, \psi].^{4)}$$

It is well-known (see [8]) that an arbitrary partially ordering R of a finite set can be extended to an ordering R'. Let R' be such an extension of the partialy ordering R of the set $\langle \mathbf{A}_1, \ldots, \mathbf{A}_k \rangle$.

Now let us consider the following automaton

$$\mathbf{A}_i^{(\gamma_i, \lambda_i)} = \mathbf{A}_i^{(\gamma_i, \lambda_i)}(X_i, A_i, Y_i, \delta_i, \lambda_i),$$

where

(3) $$X_i = \begin{cases} X & \text{if } i = 1, \\ A_1 \times \dots \times A_{i-1} \times X & \text{if } i > 1, \end{cases}$$

(4) $$Y_i = \begin{cases} A_1 \times \dots \times A_i \times X & \text{if } i < k, \\ Y & \text{if } i = k, \end{cases}$$

(5) $$\gamma_i((a_1, \ldots, a_{i-1}, x)) = \varphi_i(a_1, \ldots, a_{i-1}, x) \qquad (x \in X;\ a_j \in A_j;\ j = 1, \ldots, i-1)$$

and

(6) $$\lambda_i(a_i, (a_1, \ldots, a_{i-1}, x)) = \begin{cases} (a_1, \ldots, a_{i-1}, x) & \text{if } 1 \leqq i < k, \\ \psi(a_1, \ldots, a_k, x) & \text{if } i = k. \end{cases}$$

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4) Here we suppose that R is a partial ordering such that $\mathbf{A}_i$ is not greater than $\mathbf{A}_j$ with respect to R if $i \geqq j$. (This is possible, see [8]).

The maps γ_i in (5) are well-defined because R' is an extension of R.

Let $\mathbf{A}'=\mathbf{A}'(X, A', Y, \delta', \lambda')$ be the superposition of $A_i^{(\gamma_i, \lambda_i)}$ $(i=1, \dots, k)$, that is, $\mathbf{A}'$ is a quasi-superposition of the automata $\mathbf{A}_i$ with respect to the system $\langle X_i, Y_i, \delta_i, \lambda_i \mid i=1, \dots, k\rangle$. It is clear that $A=A'$.

We show that the maps $(\varrho_1, \varrho_2, \varrho_3)$, where

$$\varrho_1(x)=x,\ \varrho_2(a)=a,\ \varrho_3(y)=y \quad (x\in X, a\in A, y\in Y),$$

induce an isomorphism between $\mathbf{A}$ and $\mathbf{A}'$. Indeed, let $x\in X$ and $a\in A$ be arbitrary. Then

$$\varrho_2(\delta((a_1, \dots, a_k), x)) = \varrho_2(\delta_1'(a_1, \varphi_1(x)), \dots, \delta_k'(a_k, \varphi_k(a_1, \dots, a_{k-1}, x))) =$$
$$= (\delta_1'(a_1, \varphi_1(x)), \dots, \delta_k'(a_k, \varphi_k(a_1, \dots, a_{k-1}, x)))$$

and

$$\delta'(\varrho_2(a_1, \dots, a_k), \varrho_1(x)) = (\delta_1'(a_1, \gamma_1(x)), \dots, \delta_k'(a_k, \gamma_k(a_1, \dots, a_{k-1}, x))) =$$
$$= (\delta_1'(a_1, \varphi_1(x)), \dots, \delta_k'(a_k, \varphi_k(a_1, \dots, a_{k-1}, x)))$$

(because $\gamma_i(a_1, \dots, a_{i-1}, x) = \varphi_i(a_1, \dots, a_{i-1}, x)$ by virtue of (5)), that is,

$$\varrho_2(\delta((a_1, \dots, a_k), x)) = \delta'(\varrho_2(a_1, \dots, a_k), \varrho_1(x)).$$

Furthermore,

$$\varrho_3((\lambda(a_1, \dots, a_k), x)) = \varrho_3(\psi(a_1, \dots, a_k, x)) = \psi(a_1, \dots, a_k, x)$$

(where ψ is the output function of $\mathbf{A}$) and

$$\lambda'(\varrho_2(a_1, \dots, a_k), \varrho_1(x)) = \lambda'((a_1, \dots, a_k), x) =$$
$$= \lambda_k(a_k, (a_1, \dots, a_{k-1}, x)) = \psi(a_1, \dots, a_k, x),$$

the last equality being valid by virtue of (6), that is,

$$\varrho_3(\lambda(a_1, \dots, a_k), x) = \lambda'(\varrho_2(a_1, \dots, a_k), \varrho_1(x)).$$

This completes the proof of the Theorem 4.

References

[1] F. Gécseg, On R-products of automata. I, *Studia Sci. Math. Hung.*, **1** (1966), 437—441.

[2] F. Gécseg, On R-products of automata. II, *Studia Sci. Math. Hung.*, **1** (1966), 443—447.

[3] F. Gécseg, On R-products of automata. III, *Studia Sci. Math. Hung.*, **2** (1967), 163—166.

[4] W. M. Gluschkow, *Theorie der abstrakten Automaten* (Berlin, 1963).

[5] J. Hartmanis, Loop-free structure of sequential machines, *Information and Control*, **1** (1962), 25—44.

[6] Ф. Гечег, О композиции автоматов без петель, *Acta Sci. Math.*, **26** (1965), 269—272.

[7] Ф. Гечег, О группе взаимно однозначных преобразований, определенных конечными автоматами, *Кибернетика*, **1:1** (1965), 37—40.

[8] E. Szpilrajn, Sur l'extension de l'ordre partiel, *Fund. Math.*, **16** (1930), 386—389.

(Received May 25, 1968)